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# Characterizations of restricted pairs of planar graphs allowing simultaneous embedding with fixed edges ☆

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ABSTRACT

A set of planar graphs $\{G_1(V, E_1), \ldots, G_k(V, E_k)\}$ admits a *simultaneous embedding* if they can be drawn on the same pointset $P$ of order $n$ in the Euclidean plane such that each point in $P$ corresponds one-to-one to a vertex in $V$ and each edge in $E_i$ does not cross any other edge in $E_i$ (except at endpoints) for $i \in \{1, \ldots, k\}$. A *fixed edge* is an edge $(u, v)$ that is drawn using the same simple curve for each graph $G_i$ whose edge set $E_i$ contains the edge $(u, v)$. We give a necessary and sufficient condition for two graphs whose union is homeomorphic to $K_5$ or $K_{3,3}$ to admit a simultaneous embedding with fixed edges (SEFE). This allows us to characterize the class of planar graphs that always have a SEFE with any other planar graph. We also characterize the class of biconnected outerplanar graphs that always have a SEFE with any other outerplanar graph. In both cases, we provide $O(n^4)$-time algorithms to compute a SEFE.



## 1. Introduction

In many practical applications including the visualization of large graphs and very-large-scale integration (VLSI) of circuits on the same chip, edge crossings are undesirable. The same set of vertices can be used for multiple graphs where the edges of each of the graphs correspond to different colors or circuit layers. While the pairwise union of all edge sets may be non-planar, a planar drawing of each layer may be possible, as crossings between edges of distinct edge sets are permitted. However, moving one vertex to reduce crossings in one layer can introduce additional crossings in other layers. Finding such planar drawings for each layer is the basic problem of *simultaneous embedding* (SE) and this can be viewed as a generalization of the notion of planarity to multiple graphs.

Without restricting how edges common to each graph are drawn, any number of planar graphs can be drawn on the same fixed set of vertex locations [23]. However, this is no longer true if straight-line edges are required. This is the problem of *simultaneous geometric embedding* (SGE). When edges are drawn as simple curves and common edges must be drawn using the same simple curve, we have the problem of *simultaneous embedding with fixed edges* (SEFE). Since straight-line edges between a pair of vertices are also fixed edges, any graph that has a SGE also has a SEFE, but the converse is not necessarily true; see Fig. 1 that shows SGE ⊂ SEFE ⊂ SE.

☆ Preliminary reports on this topic were presented at TGGT (Fowler et al. 2008) [14] and WG (Fowler et al. 2008) [13].

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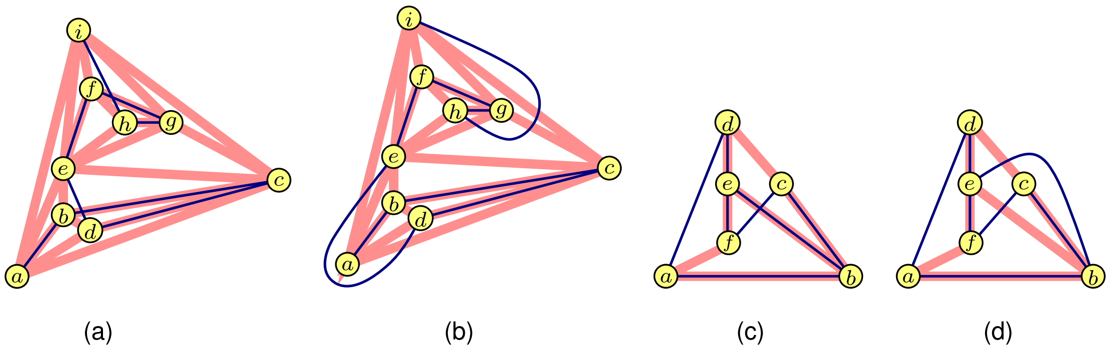


**Fig. 1.** The path and planar graph in (a) do not have a SGE with straight-line edges [2], but have a SEFE in (b). The two outerplanar graphs in (c) do not have a SEFE, but have a SE in (d) if the edge $(b, e)$ is not fixed.

Deciding whether *two* graphs have a SGE is NP-hard [11], and deciding whether *three* graphs have a SEFE is NP-complete [17]. However, determining whether two graphs can be decided to have a SEFE in polynomial-time remains an open problem. We give a necessary condition in terms of forbidden minors for pairs of planar graphs that admit a SEFE. We also characterize the class of biconnected outerplanar graphs that always have a SEFE with any other outerplanar graph. Finally, we characterize the graphs that always have a SEFE with any planar graph and show how to compute a SEFE when possible.

### 1.1. *Related work*

Any number of stars, two caterpillars (trees whose removal of all leaves yields a path) and two cycles always have a SGE, whereas three paths and two trees may not [2,18]. Which trees and which graphs always have a SGE with a path, a caterpillar, a tree, or a cycle remains unknown.

Some of these questions have been answered for the special class of unlabeled level planar (ULP) graphs, which are graphs that are level planar over all vertex labelings. ULP trees and graphs were recently characterized in terms of two forbidden tree subdivisions and five other forbidden graph subdivisions [9,15]. A graph has a SGE with any path drawn in a strictly $y$-monotone fashion if and only if the graph is ULP [10]. If edge bends are allowed, SE with two bends per edge can be found for pairs of planar graphs [6,8], while one bend per edge suffices for an outerplanar graph and a path drawn with straight-line edges [6]. For the case of fixed edges, a planar graph and a tree always have a SEFE, whereas two outerplanar graphs do not [16]. This shows that the topological problem of SEFE is less restricted than the geometric problem of SGE, given that the class of graphs that have a SEFE properly includes the class of graphs that have a SGE. This is in contrast to standard planarity in which the sets of topological and geometric planar graphs are identical [12,24,25]. Planar graphs are characterized in terms of the forbidden Kuratowski graphs, $K_5$ and $K_{3,3}$, which form two minimum examples of non-planar graphs given that $K_5$ and $K_{3,3}$ are the two smallest non-homeomorphic graphs such that removal of any edge results in a planar graph [22,26]. No similar characterization for having a SEFE in terms of forbidden pairs has been given until now, even for restricted pairs of planar graphs.

A related problem is finding the *geometric thickness* of a graph $G$, which is the minimum number of layers such that the graph can be drawn in the plane with straight-line edges where each edge is assigned to a layer so that no two edges in the same layer cross. Using simultaneous embedding techniques, it was shown that graphs of degree at most 4 have geometric thickness 2 [7].

### 1.2. *Our contribution*

1. We show there exist three paths without a SEFE. We provide a necessary and sufficient condition for the existence of a SEFE of pairs of planar graphs whose unions are a subdivided $K_5$ or $K_{3,3}$.

**Table 1**
Old and new results for pairs of graph classes that always admit a SGE or a SEFE. The shaded pairs are new.

| | SGE | | SEFE | | | | |
|---|---|---|---|---|---|---|---|
| | Path | Tree | Forest | Circular caterpillar | $K_4$ | $K_3$-multiedge | $K_3$-cycle |
| Path | ✓ [2] | ? | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] |
| Caterpillar | ✓ [2] | ? | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] |
| Tree | ? | ✗ [18] | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] | ✓ [16] |
| Outerplanar | ? | ✗ [18] | ✓ | ✓ | ✓ | ✓ | ✓ |
| Planar | ✗ [2] | ✗ [2,18] | ✓ | ✓ | ✓ | ✓ | ✗ |

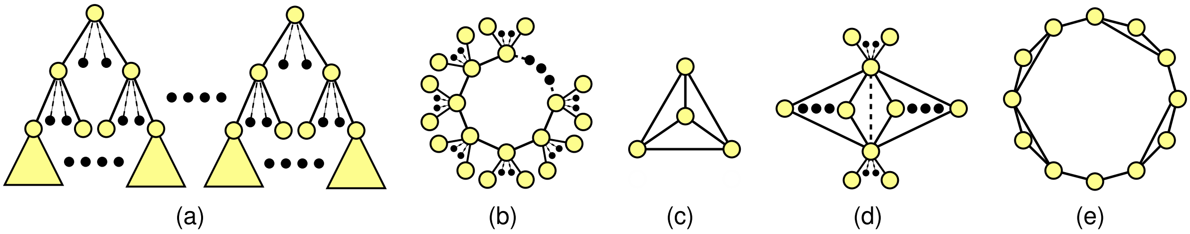


**Fig. 2.** Forests in (a), circular caterpillars (removal of degree-1 vertices yields a cycle) in (b), $K_4$ in (c), and *subgraphs* of $K_3$-multiedges (edge with any number of incident edges) in (d) have a SEFE with any planar graph. $K_3$-cycles ($n$-cycles with chords that form 3-cycles with the $n$-cycles) as in (e) have a SEFE with any outerplanar graph.

2. We characterize the class of planar graphs that have a SEFE with any planar graph to be (i) forests, (ii) circular caterpillars, (iii) $K_4$, and (iv) subgraphs of $K_3$-multiedges; see Figs.(a)–(d). We provide a similar characterization of the class of biconnected outerplanar graphs, namely $K_3$-cycles, that always have a SEFE with any outerplanar graph; see Fig. 2(e). Table 1 summarizes our results.
3. We describe $O(n^4)$-time embedding algorithms for each pair of graphs that correspond to one of the above pairs that always admits a SEFE.

### 1.3. Preliminaries

For a graph $G(V,E)$, two vertices $u$ and $v$ in $V$ are *adjacent* if the edge $(u,v)$ is in $E$. A vertex $u$ in $V$ and an edge $(v,w)$ in $E$ are *incident* if $u=v$ or $u=w$, and *non-incident*, otherwise. Likewise, two edges $e$ and $e'$ in $E$ are *incident* if they have a common endpoint in $V$. The *degree* of a vertex $v$ in $V$ is the number of edges in $E$ incident to $v$.

Let $P$ be a set of $n$ distinct points in the plane $\mathbb{R}^2$. A *planar drawing* $D$ of $G(V,E)$ with $|V|=n$ on $P$ consists of a bijection $\sigma: V \to P$ and a simple curve for each edge $(u,v) \in E$ that connects the points $\sigma(u)$ and $\sigma(v)$ in $\mathbb{R}^2$ such that the curve does not intersect any other curve except in common endpoints. Each planar drawing $D$ of $G(V,E)$ has a corresponding *planar embedding* consisting of a *combinatorial embedding* $\Gamma$, the clockwise ordering of edges in $E$ incident to each vertex in $V$ as given by $D$, and an external face $f$, also know as the *outerface*.

Let $\mathcal{G} = \{G_1(V,E_1), G_2(V,E_2), \ldots, G_k(V,E_k)\}$ be a family of $k$ graphs on $V$. $\mathcal{G}$ has a *simultaneous embedding* (SE) if there exist planar drawings of $G_i(V,E_i)$ with the same bijection $\sigma: V \to P$. If each edge is drawn with a straight-line segment, then $\mathcal{G}$ has a *simultaneous geometric embedding* (SGE). If every edge that is common to two or more graphs in $\mathcal{G}$ uses the same simple curve, then $\mathcal{G}$ has a *simultaneous embedding with fixed edges* (SEFE).

The path $(v_1,\ldots,v_k)$ is denoted by $v_1 \rightsquigarrow v_k$ (if only its endpoints $v_1$ and $v_k$ are known), by $v_1 \rightsquigarrow v_i \rightsquigarrow v_k$ (if an intermediate vertex $v_i$ for $1<i<k$ is also known), or by $v_1–v_2–\cdots–v_k$ (if all the vertices are known). Similarly, a cycle $(v_1,\ldots,v_k,v_1)$ is denoted by $v_1 \rightsquigarrow v_i \rightsquigarrow v_1$ (if an intermediate vertex $v_i$ for $1<i\leqslant k$ is known) or by $v_1–v_2–\cdots–v_k–v_1$ (if all the vertices are known). A *chain* $u \hookrightarrow v$ in a graph is a path $(u,w_1,\ldots,w_k,v)$ where two subsequent vertices are connected by an edge and the degree of every vertex $w_i$ is 2, for $i \in \{1,\ldots,k\}$.

In a graph $G(V,E)$, *subdividing* an edge $(u,v) \in E$ replaces edge $(u,v)$ with the pair of edges $(u,w)$ and $(w,v)$ in $E$ by adding $w$ to $V$. A *subdivision* of $G$ is obtained through a series of edge subdivisions. The *contraction* of an edge $(u,v)$ replaces the vertices $u$ and $v$ with the vertex $w$ that is adjacent to all the vertices that were adjacent either to $u$ or to $v$. A *minor* $H$ of $G$ is obtained through a series of edge contractions and edge deletions. A *pair* $(G_1,G_2)$ consists of two graphs with the same set of vertices where each graph contains at least one edge. A *minor pair* $(H_1,H_2)$ of the pair $(G_1,G_2)$ consists of two minors $H_1$ of $G_1$ and $H_2$ of $G_2$ each obtained by simultaneously contracting an edge in *both* graphs or deleting an edge from *either* graph. A graph $G(V,E)$ is *isomorphic* to a graph $\tilde{G}(\tilde{V},\tilde{E})$ if there exists a bijection $\mu: V \to \tilde{V}$ such that $(u,v) \in E$ if and only if $(\mu(u),\mu(v)) \in \tilde{E}$. A graph $G(V,E)$ is *homeomorphic* to a graph $\tilde{G}(\tilde{V},\tilde{E})$ if there is a subdivision of $G$ that is isomorphic to $\tilde{G}$.

## 2. Forbidden graphs that prevent simultaneous embeddings with fixed edges

We begin with Kuratowski's and Wagner's planar graph theorems [22,26].

**Theorem 1** *(Kuratowski, Wagner). A graph is non-planar if and only if it either has a subgraph homeomorphic to $K_5$ or $K_{3,3}$ or it has $K_5$ or $K_{3,3}$ as a minor.*

### 2.1. Forbidden triples of paths and cycles

Next we show that the triples without a SGE of three paths in [2] and three cycles in [1] extend to the case of SEFE.

**Theorem 2.** *There exist three paths on* 9 *vertices and three cycles on* 6 *vertices without a SEFE.*

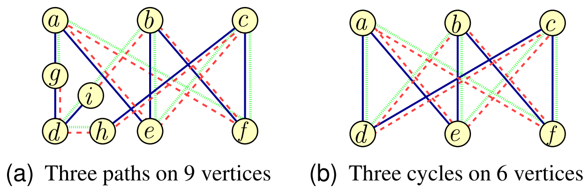


**Fig. 3.** Two graph triples without a SEFE.

**Proof.** Consider the three paths $g$–$d$–$h$–$c$–$e$–$a$–$f$–$b$–$i$, $h$–$d$–$i$–$b$–$e$–$c$–$f$–$a$–$g$, and $i$–$d$–$g$–$a$–$e$–$b$–$f$–$c$–$h$ and the three cycles $a$–$d$–$c$–$f$–$b$–$e$–$a$, $a$–$e$–$c$–$d$–$b$–$f$–$a$, and $a$–$f$–$c$–$e$–$b$–$d$–$a$ shown in Fig. 3. In both cases, the union forms a subdivided $K_{3,3}$ and any drawing must have a crossing by Theorem 1. Each edge in the union belongs to two paths (or cycles). Such a crossing must be between two pairs of paths (or cycles). Since there are only three paths (or cycles) and the drawing has fixed edges, one path (or cycle) must self-intersect. $\square$

This shows that the set of triples that admit a SEFE is fairly restricted. As a result, we can focus our attention on forbidden pairs of graphs without a SEFE.

### 2.2. Minimal forbidden pairs

Suppose a pair $G_1$ and $G_2$ does not have a SEFE as in Fig. 4(a). If deleting any edge from either graph allows a SEFE, then $G_1$ and $G_2$ are *edge minimal* as in Fig. 4(b). If a degree-2 vertex $v$ (adjacent to $u$ and $w$) in the union of $G_1$ and $G_2$ is not a degree-1 vertex in either $G_1$ or $G_2$, then we can *unsubdivide* the vertex by deleting $v$ and replacing edges $(u,v)$ and $(v,w)$ with the edge $(u,w)$ in $G_1$ and/or $G_2$.[3] A pair of graphs for which this can no longer be done is *vertex minimal* as in Fig. 4(c). A *minimal forbidden pair* is a pair that is edge and vertex minimal and does not admit a SEFE. Let $G_1 \cup G_2$ and $G_1 \cap G_2$ denote the *union* and the *intersection* of pair $G_1(V,E_1)$ and $G_2(V,E_2)$ with edge sets $E_1 \cup E_2$ and $E_1 \cap E_2$, respectively; see Figs.(c)–(d). Suppose then that $G_1 \cup G_2$ is homeomorphic to a graph $G$ with no degree-2 vertices. Let $u \hookrightarrow v$ in $G_1 \cup G_2$ be the chain corresponding to the edge $(u,v)$ in $G$. Chain $u \hookrightarrow v$ *is incident to chain* $x \hookrightarrow y$ in $G_1 \cup G_2$ if and only if edge $(u,v)$ is incident to edge $(x,y)$ in $G$. An *alternating chain* is a maximal chain $u \hookrightarrow v$ in which the edges strictly alternate between being in either $G_1$ or $G_2$.[4] The *alternating chain subgraph*, $G_1 \uplus G_2$, is the subgraph of $G_1 \cup G_2$ consisting only of alternating chains; see Fig. 4(e). An *exclusive edge* is an edge $(u,v)$ that is only in $G_1$ or $G_2$, while an *inclusive edge* is an edge $(u,v)$ in $G_1 \cap G_2$. The *exclusive edge subgraph* of $G_1$, $G_1 \setminus G_2$, is the subgraph of $G_1 \cup G_2$ consisting of exclusive edges from $G_1$, where $G_2 \setminus G_1$ is defined analogously; see Figs.(f)–(g).

Next we define a transformation of a pair of graphs that can reduce the number of edges and vertices in the pair.

**Definition 3.** A pair $(G_1, G_2)$ can be *reduced* to the pair $(G_1', G_2')$ by *reducing* each maximal chain $u \hookrightarrow v$ in $G_1 \cup G_2$ to either an inclusive edge, an exclusive edge, or an alternating chain as follows:

1. If $u \hookrightarrow v$ is in $G_1 \cap G_2$, then replace $u \hookrightarrow v$ with the inclusive edge $(u,v)$ in both $G_1$ and $G_2$.
2. If chain $u \hookrightarrow v$ is in $G_i$ but is missing edges in $G_j$ for $i \neq j$, then replace $u \hookrightarrow v$ with the single exclusive edge $(u,v)$ in $G_i$ and remove any edges of $u \hookrightarrow v$ from $G_j$.
3. If chain $u \hookrightarrow v$ is missing one or more edges from both $G_1$ and $G_2$, then for each edge $e$ in $u \hookrightarrow v$ that is in $G_1 \cap G_2$, contract the edge $e$ so that no edge in $u \hookrightarrow v$ remains in $G_1 \cap G_2$. Next, replace each maximal subchain $x \hookrightarrow y$ of $u \hookrightarrow v$ that is in $G_i$ for $i \in \{1,2\}$ with an edge $(x,y)$ in $G_i$ so that $u \hookrightarrow v$ now forms an alternating chain.

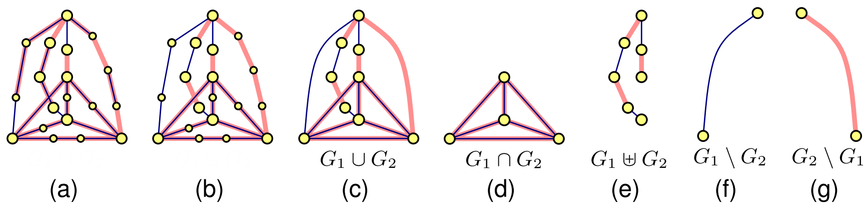


**Fig. 4.** Removing extraneous edges from (a) that do not affect whether a pair of graphs has a SEFE gives (b). Unsubdividing degree-2 vertices in (b) gives (c), the union $G_1 \cup G_2$, that can be partitioned into the four subgraphs in (d)–(g), namely, the intersection $G_1 \cap G_2$, the alternating chain subgraph $G_1 \uplus G_2$, and the two exclusive edge subgraphs $G_1 \setminus G_2$ and $G_2 \setminus G_1$, respectively.

[3] If the edges $(u,v)$ and $(v,w)$ belong to both $G_1$ and $G_2$, then $(u,w)$ belongs to both $G_1$ and $G_2$, and if the edges $(u,v)$ and $(v,w)$ both belong only to $G_1$ or $G_2$, then $(u,w)$ only belongs to that graph.

[4] Recall that a chain consists of a sequence of two or more edges connected by degree-2 vertices. Hence, an alternating chain must have at least one edge that belongs only to $G_1$ and at least one other edge that belongs only to $G_2$.

A *reduced pair* is a pair such that every chain in $G_1 \cup G_2$ is either an exclusive edge, an inclusive edge, or an alternating chain.

The next lemma shows that reducing a pair does not affect the possibility of whether the pair admits a SEFE.

**Lemma 4.** *The pair $(G_1, G_2)$ has a SEFE if and only if its reduced pair $(G'_1, G'_2)$ has a SEFE.*

**Proof.** Let $(G_1^k, G_2^k)$ be a sequence of $n+1$ pairs for $0 \leqslant k \leqslant n$ such that $(G_1^0, G_2^0)$ is the initial pair $(G_1, G_2)$ and $(G_1^n, G_2^n)$ is the final reduced pair $(G'_1, G'_2)$. Further suppose that the pair $(G_1^{k+1}, G_2^{k+1})$ is obtained by performing exactly one of the three reductions of Definition 3 on some maximal chain $u \hookrightarrow v$ in $G_1^k \cup G_2^k$ for some $0 \leqslant k < n$.

We first assume that a SEFE exists for $(G_1^k, G_2^k)$. We show how to construct a SEFE for $(G_1^{k+1}, G_2^{k+1})$. After performing the reduction of Definition 3, the crossing-free path taken by $u \hookrightarrow v$ in the SEFE of $(G_1^k, G_2^k)$ is used to draw the corresponding edge $(u, v)$ or alternating chain $u \hookrightarrow v$ of $(G_1^{k+1}, G_2^{k+1})$. The remainder of $G_1^{k+1}$ and $G_2^{k+1}$ is drawn in exactly the same manner as $G_1^k$ and $G_2^k$. Given that $(G_1^{k+1}, G_2^{k+1})$ is the combination of replacing chains with single edges and/or the removal or the contraction of extraneous edges from $(G_1^k, G_2^k)$, the resulting simultaneous drawing of $(G_1^{k+1}, G_2^{k+1})$ must also be a SEFE.

Next assume that a SEFE exists for $(G_1^{k+1}, G_2^{k+1})$. We reverse the above procedure to obtain a simultaneous drawing of $(G_1^k, G_2^k)$. Replacing a single edge with a chain in the same graph does not lead to any crossings. However, placing an additional edge in $G_1^{k+1}$ and/or $G_2^{k+1}$ (when reversing the second or third reduction of Definition 3) can. To avoid this problem, any extra edge in $G_1^k$ and/or $G_2^k$ must be placed along the path of the corresponding edge $(u, v)$ or the alternating chain $u \hookrightarrow v$ in $(G_1^{k+1}, G_2^{k+1})$ so that it occurs strictly between any two crossings involving $(u, v)$ or $u \hookrightarrow v$, respectively. Given that this is always possible, this procedure produces a SEFE of $(G_1^k, G_2^k)$. $\square$

**Observation 5.** *A minimal forbidden pair $(G_1, G_2)$ is a reduced pair of planar graphs.*

**Proof.** Consider a maximal chain $u \hookrightarrow v$ in a minimal forbidden pair. If $u \hookrightarrow v$ were in $G_1$ and/or $G_2$, then $u \hookrightarrow v$ could be reduced to either an inclusive or exclusive edge $(u, v)$. Since this would not affect whether the pair admits a SEFE by Lemma 4, this would violate the vertex minimality of the pair. Hence, $u \hookrightarrow v$ must have at least one edge missing from both $G_1$ and $G_2$. However, if $u \hookrightarrow v$ is not already an alternating chain, then it could be reduced to one without affecting whether the pair admits a SEFE, again by Lemma 4. In the process, either an edge would be removed violating the edge minimality of the pair or subchains of $u \hookrightarrow v$ are replaced with single edges violating the vertex minimality of the pair. Hence, $u \hookrightarrow v$ must be an alternating chain. Every other edge not in a chain must either be an inclusive or exclusive edge. Thus, $(G_1, G_2)$ is a reduced pair.

Next, suppose that $G_1$ (or $G_2$) is non-planar so that the pair does not admit a SEFE. For such a pair to be edge minimal, $G_1$ (or $G_2$) must be minimally non-planar and $G_2$ (or $G_1$) must be empty. Otherwise, an edge could be removed from one of the pair of graphs without affecting whether the pair admits a SEFE. However, by definition each graph must contain at least one edge in order to form a valid pair. As a result, both $G_1$ and $G_2$ must be planar. $\square$

Suppose $(G_1, G_2)$ is a reduced pair. We observe that the edges of $G_1 \cup G_2$ are partitioned into $G_1 \cap G_2$, $G_1 \uplus G_2$, $G_1 \setminus G_2$, and $G_2 \setminus G_1$; see Figs.(d)–(g). Next we show that we only need to consider crossings between non-incident edges.

**Observation 6.** *Crossings between incident edges in a non-planar drawing can be removed without affecting the crossings of non-incident edges.*

**Proof.** This can be done by interchanging the simple curves of a pair of incident edges from their common vertex to their first intersection point $p$. Separating the curves at $p$ by a small distance eliminates the crossing without affecting the rest of the drawing. Repeating this process eliminates all of the crossings of incident edges. $\square$

Hence, we only need to consider crossings of non-incident edges. Removing an edge from either of the Kuratowski subgraphs $K_5$ or $K_{3,3}$ of Theorem 1 allows a planar embedding. Only one crossing needs to be introduced when re-inserting the edge, since any pair of vertices that do not share a common face are separated by at most one edge in the embedding. Moreover, consider any drawing of $K_5$ (or $K_{3,3}$) with only one crossing. Then, one can suitably give name to the vertices so that only a given pair of edges cross, which with Observation 6 gives the next corollary.

**Corollary 7.**

(a) *Every drawing of $K_5$ and $K_{3,3}$ has a crossing between non-incident edges.*
(b) *$K_5$ and $K_{3,3}$ can be drawn with only one crossing between any pair of non-incident edges.*

We use this corollary to produce a sufficient condition for two graphs to admit a SEFE.

**Lemma 8.** *Suppose the union $G_1 \cup G_2$ of a reduced pair $(G_1, G_2)$ is homeomorphic to $K_5$ or $K_{3,3}$. Let $u \hookrightarrow v$ and $x \hookrightarrow y$ be non-incident chains that are in $G_1 \cup G_2$ but not in $G_1 \cap G_2$. If either chain belongs to $G_1 \uplus G_2$ or one belongs to $G_1 \setminus G_2$ and the other belongs to $G_2 \setminus G_1$, then $G_1$ and $G_2$ have a SEFE.*

**Proof.** By Corollary 7(b), $K_5$ or $K_{3,3}$ can always be drawn so that only $(u, v)$ and $(x, y)$ cross. Hence, there is a SEFE in which an alternating chain in $G_1 \uplus G_2$ only crosses an edge in either $G_1 \setminus G_2$ or $G_2 \setminus G_1$. Likewise, an edge in $G_1 \setminus G_2$ can cross any non-incident edge in $G_2 \setminus G_1$. □

Using Lemma 8, we next determine whether a reduced pair of graphs whose union is homeomorphic to $K_5$ or $K_{3,3}$ can admit a SEFE.

**Corollary 9.** *Suppose the union $G_1 \cup G_2$ of a reduced pair $(G_1, G_2)$ is homeomorphic to $K_5$ or $K_{3,3}$. The pair $(G_1, G_2)$ has no SEFE if and only if*

(i) *every non-incident edge of an alternating chain in $G_1 \uplus G_2$ is in $G_1 \cap G_2$ and*
(ii) *every non-incident edge of an exclusive edge in $G_1 \setminus G_2$ is in $G_1$.*

**Proof.** For necessity, suppose the pair $(G_1, G_2)$ does not have a SEFE. Consider a chain $x \hookrightarrow y$ in $G_1 \cup G_2$ but not in $G_1 \cap G_2$ that is not incident to an alternating chain $u \hookrightarrow v$ in $G_1 \uplus G_2$. By Lemma 8, the pair $(G_1, G_2)$ would have a SEFE since the chain $u \hookrightarrow v$ is in $G_1 \uplus G_2$ and neither chain is in $G_1 \cap G_2$. Next consider a chain $x \hookrightarrow y$ in $G_1 \cup G_2$ but not in $G_1$ that is not incident to an exclusive edge $(u, v)$ in $G_1 \setminus G_2$. Again by Lemma 8, the pair $(G_1, G_2)$ would have a SEFE since the chain $x \hookrightarrow y$ either is in $G_1 \uplus G_2$ or is in $G_2 \setminus G_1$.

For sufficiency, suppose conditions (i) and (ii) hold. Since the union forms a subdivided $K_5$ or $K_{3,3}$, by Corollary 7(a) at least one pair of non-incident chains $u \hookrightarrow v$ and $x \hookrightarrow y$ cross. If either chain is in $G_1 \cap G_2$, then there must be a crossing in $G_1$ or $G_2$. If either chain is in $G_1 \uplus G_2$, then by (i) the other would be in $G_1 \cap G_2$, again giving a crossing in $G_1$ or $G_2$. If either chain is in $G_i \setminus G_j$, for $i \neq j$ and $i, j \in \{1, 2\}$, then by (ii) the other chain would be in $G_i$, thus implying a crossing. Hence, $G_1$ and $G_2$ cannot have a SEFE. □

**Theorem 10.** *There are* 17 *minimal forbidden pairs with a union homeomorphic to $K_5$ or $K_{3,3}$.*

**Proof.** Let $G_{i,1}$ and $G_{i,2}$ denote one of the 17 pairs for $i \in \{1, \ldots, 17\}$ in Figs. 5 and 6. The union of the first eleven pairs is homeomorphic to $K_5$, while the union of the remaining six pairs is homeomorphic to $K_{3,3}$. One can verify that all the non-incident edges of any alternating chain are in $G_1 \cap G_2$ and every edge non-incident to an exclusive edge of $G_{i,1}$ (or $G_{i,2}$) is also in $G_{i,1}$ (or $G_{i,2}$). This satisfies Corollary 9 implying that none of these pairs has a SEFE.

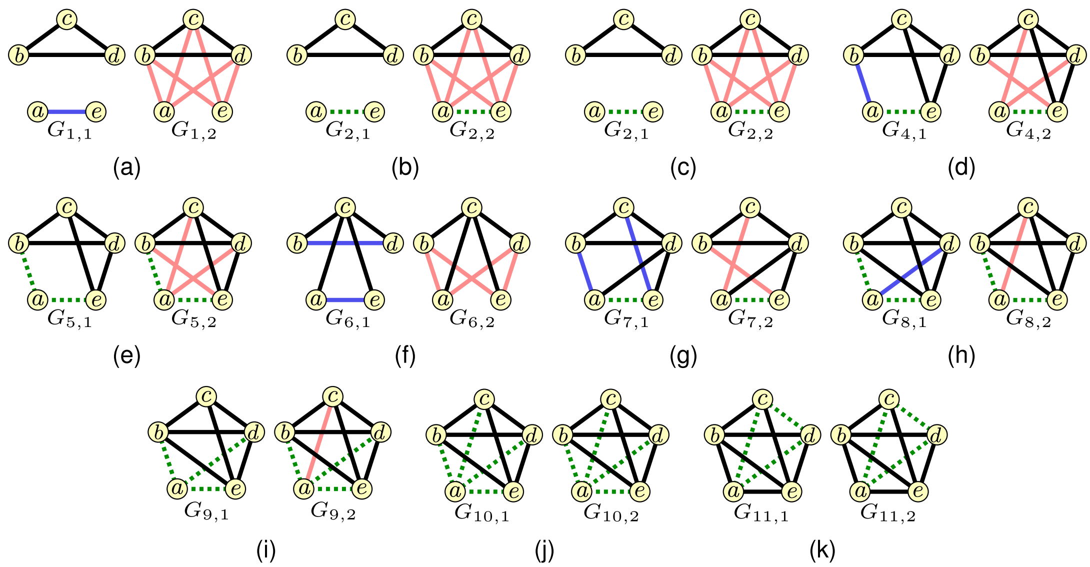


**Fig. 5.** Eleven $K_5$ minimal forbidden pairs. Solid light grey (or pink, if in color) and medium grey (or blue, if in color) lines are exclusive edges. Solid black lines common to each pair are inclusive edges. Dashed lines are alternating chains.

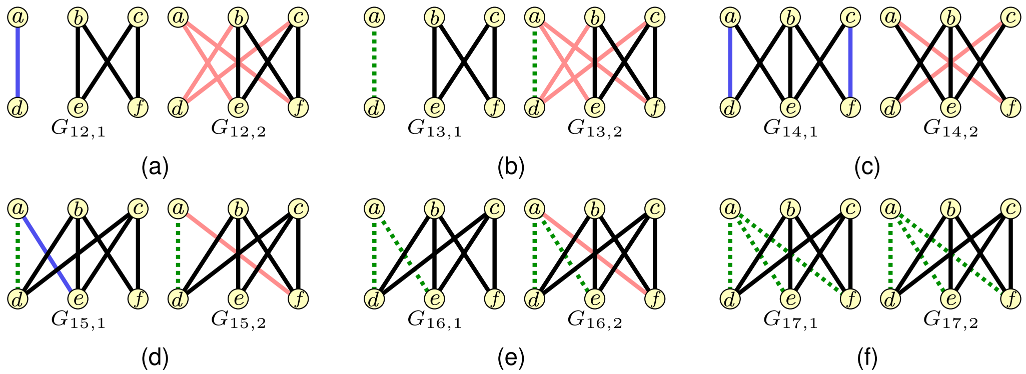


**Fig. 6.** Six $K_{3,3}$ minimal forbidden pairs. Solid light grey (or pink, if in color) and medium grey (or blue, if in color) lines are exclusive edges. Solid black lines common to each pair are inclusive edges. Dashed lines are alternating chains.

Next, we show that the 17 pairs are minimal. Since the only chains in any of the pairs are alternating chains, each pair is a reduced pair, and hence, any edge $e$ in pair $(G_{i,1}, G_{i,2})$ is either (i) an edge of an alternating chain, (ii) an exclusive edge, or (iii) an inclusive edge. Removing edge $e$ in cases (i) or (ii) means that the union is no longer homeomorphic to $K_5$ or $K_{3,3}$. Each inclusive edge $(u, v)$ in $(G_{i,1}, G_{i,2})$ is such that (a) there exists an exclusive edge in $G_{i,1}$ not incident to $(u, v)$ or (b) there exists an alternating chain not incident to $(u, v)$. In fact, if all the edges not incident to $(u, v)$ belonged to $G_{i,1}$, then the edge $(u, v)$ could be removed from $G_{i,2}$, and the pair would still have no SEFE. However, upon direct examination of the 17 pairs, this is never the case. Hence, removing the inclusive edge $e$ in case (iii) implies that the conditions of Corollary 9 are no longer satisfied. Therefore, all 17 forbidden pairs are minimal.

We next show that our 17 pairs are the only minimal forbidden pairs homeomorphic to $K_5$ or $K_{3,3}$. We do this by exhaustively considering all minimal forbidden pairs $(G_1, G_2)$ on five and six vertices (not including degree-2 vertices), respectively, that have enough edges to force a crossing. When constructing such pairs, we have four types of chains or edges to connect vertices that are adjacent in the $K_5$ or $K_{3,3}$: exclusive edges in $G_1$ or $G_2$, inclusive edges in $G_1 \cap G_2$, and alternating chains in $G_1 \uplus G_2$.

We use the following procedure to construct a minimal forbidden pair that is homeomorphic to $K_5$ or $K_{3,3}$:

(a) Add zero or more exclusive edges to $G_1$ and zero or more alternating chains to $G_1$ and $G_2$.
(b) For each edge/chain $x$ added in step (a):
  (1) If $x$ is an exclusive edge, then add all non-incident edges of $x$ in the $K_5$ or the $K_{3,3}$ homeomorphic union not yet existing in $G_1$ as inclusive edges to $G_1$ and $G_2$.
  (2) If $x$ is an alternating chain, then add all non-incident edges of $x$ in the $K_5$ or the $K_{3,3}$ homeomorphic union as inclusive edges to $G_1$ and $G_2$.
(c) All edges in the $K_5$ or the $K_{3,3}$ homeomorphic union not in $G_1$ are added to $G_2$.

Afterward, both conditions of Corollary 9 hold and the two graphs have no SEFE. Further, we can assume w.l.o.g. that $G_2$ has at least as many exclusive edges as $G_1$.

We delineate the cases by the number of exclusive edges added to $G_1$ and/or alternating chains added to $G_1 \uplus G_2$:

*No exclusive edge/alternating chain added in step* (a) *and the union is homeomorphic to $K_5$ or $K_{3,3}$*: In this case, $G_1$ would be empty forcing $G_2$ to be homeomorphic to $K_5$ or $K_{3,3}$. However, by Observation 5 both graphs must be planar in order for $(G_1, G_2)$ to form a minimal forbidden pair.

*One exclusive edge/alternating chain added in step* (a) *and the union is homeomorphic to $K_5$ or $K_{3,3}$*: Pairs $(G_{1,1}, G_{1,2})$, $(G_{2,1}, G_{2,2})$, $(G_{12,1}, G_{12,2})$, and $(G_{13,1}, G_{13,2})$ are the only possibilities in which there is one exclusive edge in $G_1$ or one alternating chain in $G_1 \uplus G_2$.

*Two non-incident exclusive edges/alternating chains added in step* (a) *and the union is homeomorphic to $K_5$ or $K_{3,3}$*: Adding either two non-incident alternating chains or an alternating chain and a non-incident exclusive edge would violate Corollary 9. The other case of two non-incident edges that are exclusive in $G_1$ is given by pairs $(G_{6,1}, G_{6,2})$ and $(G_{14,1}, G_{14,2})$.

*Two incident exclusive edges/alternating chains added in step* (a) *and the union is homeomorphic to $K_5$*: Pairs $(G_{3,1}, G_{3,2})$, $(G_{4,1}, G_{4,2})$, and $(G_{5,1}, G_{5,2})$ give the three possibilities of two incident chains that are exclusive and/or alternating where $G_1 \cup G_2$ is homeomorphic to $K_5$.

*Two incident exclusive edges/alternating chains added in step* (a) *and the union is homeomorphic to $K_{3,3}$*: If $G_1$ contains two incident exclusive edges incident to a vertex $u$, then after adding their seven non-incident edges to $G_1 \cap G_2$ either (a) $G_2 \setminus G_1$ would only contain an exclusive edge $(u, v)$ or (b) $G_1 \uplus G_2$ would contain an alternating chain $u \hookrightarrow v$. In either case, $G_1$ would contain one or two more exclusive edges than $G_2$ violating the assumption that $G_2$ contains at least as many exclusive edges as $G_1$. Pairs $(G_{15,1}, G_{15,2})$ and $(G_{16,1}, G_{16,2})$ give the remaining two possibilities of two incident edges/chains such that at least one is alternating where $G_1 \cup G_2$ is homeomorphic to $K_{3,3}$.

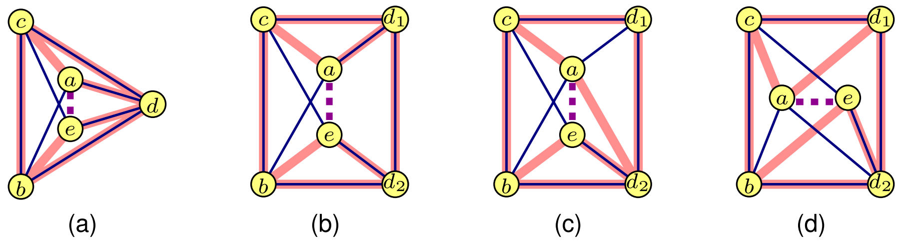


**Fig. 7.** The pair $(G_{7,1}, G_{7,2})$ in (a) is a minor pair of the two forbidden pairs in (b) and (c), which have no SEFE, as well as the pair in (d), which has the given SEFE.

*Three or more exclusive edges/alternating chains added in step* (a) *and the union is homeomorphic to* $K_5$: At most two edges/chains can be non-incident if $G_1 \cup G_2$ is homeomorphic to $K_5$, hence at least two edges/chains that are added must be all incident to the same vertex. However, if two or more of such incident edges are exclusive edges, then after adding all their non-incident edges to $G_1 \cap G_2$, $G_2$ would have at most one exclusive edge, violating the assumption that $G_2$ contains at least as many exclusive edges as $G_1$. Hence, either $G_1$ contains two non-incident exclusive edges as in pair $(G_{7,1}, G_{7,2})$, one exclusive edge as in $(G_{8,1}, G_{8,2})$, or no exclusive edges. In the last case, either $G_1 \uplus G_2$ contains three or four alternating chains that are all incident as in pairs $(G_{9,1}, G_{9,2})$ and $(G_{10,1}, G_{10,2})$ or three alternating chains that are only pairwise incident as in pair $(G_{11,1}, G_{11,2})$.

*Three or more exclusive edges/alternating chains added in step* (a) *and the union is homeomorphic to* $K_{3,3}$: If two of the edges/chains are non-incident, then both must be an exclusive edge in $G_1$ since having two non-incident alternating chains would violate Corollary 9. However, after adding all their non-incident edges to $G_1 \cap G_2$, one has pair $(G_{14,1}, G_{14,2})$. Since $G_2$ must have at least as many exclusive edges as $G_1$ by assumption, there cannot be a third exclusive edge added to $G_1$ or an alternating chain added to $G_1 \uplus G_2$. Hence, all the added edges/chains must be incident, and exactly three edges/chains were added since $K_{3,3}$ has maximum degree of 3. As previously noted, $G_1$ cannot have two incident exclusive edges when $G_1 \cup G_2$ is homeomorphic to $K_{3,3}$, since $G_2$ would not have as many exclusive edges as $G_1$. Hence, $G_1$ cannot have an exclusive edge (otherwise $G_1$ would become homeomorphic to $K_{3,3}$ after adding all the non-incident edges in step (b), and hence, non-planar), which leaves $(G_{17,1}, G_{17,2})$ as the only remaining possibility. □

Unlike standard planar graphs in which the set of forbidden minors is identical to the set of forbidden subdivisions by Theorem 1, the same is not true for graphs that admit a SEFE. Fig. 7 shows three pairs with the same minor pair $(G_{7,1}, G_{7,2})$ in Fig. 7(a). Each pair is obtained by "uncontracting" vertex $d$ to form the fixed edge $(d_1, d_2)$ in Figs. 7(b)–(d). Figs. 7(b)–(c) are forbidden pairs, whereas, Fig. 7(d) is not. Figs. 7(c)–(d) are examples in which a new fixed edge $(a, d)$ is created from the exclusive edges $(a, d_1)$ in $G_1 \setminus G_2$ and $(a, d_2)$ in $G_2 \setminus G_1$ by contracting edge $(d_1, d_2)$ to vertex $d$ in Fig. 7(a). To avoid this, we define a *fixed edge minor pair* as a minor pair $(H_1, H_2)$ of $(G_1, G_2)$ that is obtained by only contracting edges in which no new fixed edges are created. A fixed edge minor pair of the graph pair in Fig. 7(b) is shown in Fig. 7(a).

This leads to the following observation.

**Observation 11.** *Pair $(G_1, G_2)$ cannot have a SEFE if it has a fixed edge minor pair $(H_1, H_2)$ that has no SEFE.*

This observation then allows us to extend Theorem 10 to the next corollary.

**Corollary 12.** *Pair $(G_1, G_2)$ has no SEFE if the pair has a fixed edge minor pair $(H_1, H_2)$ that is isomorphic to one of the* 17 *minimal forbidden pairs of Theorem* 10*.*

This forms a necessary condition for when two graphs admit a SEFE, but is insufficient since Fig. 7(c) does not have a SEFE, nor does it have any of the 17 fixed edge minor pairs.

## 3. Characterizing pairs of planar graphs having a simultaneous embedding with fixed edges

We next determine the graphs that *always* have a SEFE with *any* planar graph and describe algorithms for creating such SEFE drawing. Let $\mathcal{P}$ be the set of planar graphs and $\mathcal{P}_{\mathsf{SEFE}}$ be the subset of $\mathcal{P}$ containing forests, *circular caterpillars* (removal of all degree-1 vertices yields a cycle), $K_4$, and the subgraphs of $K_3$*-multiedges* (edge $(x, y)$ with the incident edges $(x, z)$ and/or $(y, z)$ for each $z \in V \setminus \{x, y\}$).

**Lemma 13.** *A graph $G$ is in $\mathcal{P}_{\mathsf{SEFE}}$ if and only if $G$ does not contain a subgraph homeomorphic to $G_{1,1}$.*

**Proof.** First, we show necessity. Let $G \in \mathcal{P}_{\mathsf{SEFE}}$ and let $H$ be the graph consisting of a subdivision of $G_{1,1}$, i.e. a $K_3$ and a disjoint edge. A forest has no cycles unlike $H$. While a circular caterpillar has a cycle, all the other edges are incident to the cycle. A $K_4$ has four vertices, whereas, $H$ has at least five. Finally, every subgraph of a $K_3$-multiedge containing a cycle has

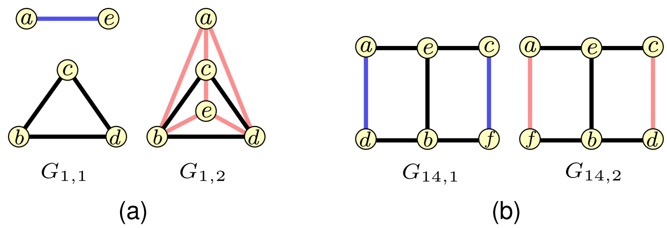


**Fig. 8.** Planar drawings for the pairs $(G_{1,1}, G_{1,2})$ and $(G_{14,1}, G_{14,2})$.

a 3-cycle, $x \rightsquigarrow y \rightsquigarrow z \rightsquigarrow x$, if the subgraph contains the edge $(x, y)$, or a 4-cycle, $x \rightsquigarrow z_1 \rightsquigarrow y \rightsquigarrow z_2 \rightsquigarrow x$, otherwise. Regardless, every edge in the subgraph is in the cycle or is incident to $x$ or $y$.

Next we show sufficiency. Let $\tilde{G} \in \mathcal{P} \setminus \mathcal{P}_{\mathsf{SEFE}}$. It suffices to show that $\tilde{G}$ has a subgraph homeomorphic to $H$. The graph $\tilde{G}$ must have a cycle since otherwise it would be a forest. Let $C$ be a cycle in $\tilde{G}$ of maximum length, and let $e$ be any edge in $\tilde{G} \setminus C$ that either (i) is not incident to $C$ or (ii) is a chord of $C$. If there is no such edge, then $\tilde{G}$ is either a circular caterpillar or a subgraph of a $K_3$-multiedge. In case (i), the graph $\tilde{G}$ contains a subgraph homeomorphic to $H$. In case (ii), let $(x, y)$ be the chord of $C$ and let $x \rightsquigarrow y$ and $y \rightsquigarrow x$ be the two paths that form $C$. Further, let $C'$ and $C''$ be the two cycles consisting of the chord $(x, y)$ and the paths $x \rightsquigarrow y$ and $y \rightsquigarrow x$, respectively. If $C'$ (or $C''$) is a $k$-cycle for some $k \geqslant 4$, then the cycle $C''$ (or $C'$) has a non-incident edge in $C$ so that $\tilde{G}$ would contain a subgraph homeomorphic to $H$. Thus, in this case $C'$ and $C''$ must both be 3-cycles and $C$ must be a 4-cycle.

Hence, all cycles in $\tilde{G}$ are 3-cycles or 4-cycles where $C$ is a 4-cycle (since it must have chord $e$). If $C$ and some other cycle $C'$ only share a vertex or a single edge, then $C'$ would have a non-incident edge in $C$. Hence, $C$ and $C'$ either form $K_4$ or must share two non-adjacent vertices $x$ and $y$. If $C$ and $C'$ form $K_4$, then either $\tilde{G}$ contains other edges (and then $\tilde{G}$ contains a subgraph homeomorphic to $H$), or $\tilde{G}$ is a $K_4$. Thus, all the 4-cycles share two non-adjacent vertices $x$ and $y$. Furthermore, all 3-cycles have the common edge $(x, y)$ if it exists. Any non-cycle edge $e$ must be incident to all the cycles implying that $e$ is either $(x, z)$ or $(y, z)$ for some vertex $z$ of degree 1 ($z$ would be part of cycle if its degree was greater than 1). Thus, if $\tilde{G}$ has multiple cycles but is not a $K_4$, then $\tilde{G}$ is a subgraph of some $K_3$-multiedge. Finally, if $C$ is the only cycle, then all the vertices not in $C$ have degree 1 so that $\tilde{G}$ is a circular caterpillar. □

Together Corollary 12 and Lemma 13 allow us to determine when a graph can have a SEFE with any planar graph with the following lemma:

**Lemma 14.** *If $G$ is a planar graph not in $\mathcal{P}_{\mathsf{SEFE}}$, then there is a planar graph $G'$ that does not admit a SEFE with $G$.*

**Proof.** Let $G \in \mathcal{P} \setminus \mathcal{P}_{\mathsf{SEFE}}$ and $G' \in \mathcal{P}$ such that $G'$ contains a subgraph homeomorphic to $G_{1,2}$. In all the 17 pairs of Theorem 10, both graphs have a subgraph homeomorphic to $G_{1,1}$, which is a $K_3$ and a disjoint edge; see Fig. 8(a). By Lemma 13, we know that $G$ contains a subgraph homeomorphic to $G_{1,1}$. Thus, $(G, G')$ cannot have a SEFE by Corollary 12 since $G'$ contains a subgraph homeomorphic to $G_{1,2} \in \mathcal{P}$. □

This shows that $G \in \mathcal{P}_{\mathsf{SEFE}}$ is a necessary condition for $G$ to have a SEFE with any planar graph. To show that this condition is also sufficient, we need to show how to compute a SEFE for each pair $(G, G')$, where $G \in \mathcal{P}_{\mathsf{SEFE}}$ and $G' \in \mathcal{P}$. Before we do, we give the high-level algorithm for computing a SEFE in which each edge is drawn in sequential order.

First, we need a general drawing algorithm in which we can specify an arbitrary pointset with existing line segments (representing straight-line edges already in the drawing) that we can augment by drawing each remaining edge using at most a linear number of bends per edge. This is possible provided that there exists a combinatorial embedding of the final planar drawing that is compatible with the existing planar embedding of the partial drawing.

Unfortunately, determining whether a compatible embedding of a planar graph always exists given only a partial embedding of a subgraph is an open problem [20]. Luckily, all of the graphs we wish to draw are relatively simple in nature. This allows us to provide an alternate input from which a compatible combinatorial embedding can be derived, namely, an order in which to draw the remaining edges along with their corresponding faces (in the updated drawings).

To further simplify our drawing algorithm, we will require for edges connecting any disconnected components to be drawn before other edges. As a result, any edge thereafter only connects two vertices in same connected component. This avoids having to route such edges around disconnected components (or isolated vertices) in anticipation of some future edge that would later connect those components (or isolated vertices) to the rest of the graph.

**Lemma 15.** *An $n$-vertex planar graph $G(V, E)$ can be drawn in $O(n^4)$ time on any fixed pointset $P$ in general position of size $n$ for a given bijection $\sigma : V \to P$ where* (i) *a subset of edges $E'$ of $E$ have already been drawn as straight-line non-crossing edges* (*except at common endpoints*), (ii) *each remaining edge of $E \setminus E'$ is drawn in a given sequential order with at most $O(n)$ bends, where edges connecting disconnected components precede any edge connecting two vertices in the same component, and* (iii) *the corresponding face into which to draw each edge is given so that the resulting combinatorial embedding $\Gamma$ of $G$ is compatible with the given partial planar embedding of $G'(V, E')$.*

**Algorithm:** DRAW-FIXED-EDGE

▷ Given a planar graph $G'(V, E')$ on $n$ vertices corresponding to $n$ points in general position and a set of $O(n^2)$ line segments forming fixed edges of $E'$, determine a path $p$ corresponding to an edge $(u, v) \notin E'$ in a face $f$ of $G'$ in $O(n^3)$ time such that no part of $p$ is within a distance $\varepsilon > 0$ from any other points or line segments.

**Input**: Planar graph $G'(V, E')$ on $n$ vertices, set of $n$ points $P$ in $\mathbb{R}^2$, bijection $\sigma : V \to P$, set of $O(n^2)$ line segments $L$, surjection $\rho : L \to E'$, edge $(u, v) \notin E'$, and face $f$ of $G'$

**Output**: Path $p$ corresponding to the edge $(u, v)$ with $O(n)$ bends in face $f$ of $G'$ that has a minimal distance $\varepsilon > 0$ from any other points in $P$ or line segments in $L$

**if** $(u, v) \in E'$ **or** $f \notin G'$ **or** $u, v \notin f$ **then return** $\varnothing$
$Q \leftarrow P$, **foreach** $\ell \in L$ **do** $Q \leftarrow Q \cup \{p, q : p, q$ are endpoints of $\ell\}$
Using $\rho$, determine which segments of $L$ correspond to edges of the face $f$, and let $R$ be the region enclosed by these segments.
Using $L$ as input, determine a Euclidean shortest path $p$ from $\sigma(u)$ to $\sigma(v)$ confined to $R$ in $O(|Q| \log |Q|)$ time.
Let $B := \{b_0, \ldots, b_{k+1}\} \subseteq Q$ such that $p$ encounters $b_i$ before $b_j$ for $i < j$ where $b_0 \leftarrow \sigma(u)$ and $b_{k+1} \leftarrow \sigma(v)$.
Define $dist(p, \ell)$ to be the smallest distance between $p$ and $\ell$ where $p$ is a point and $\ell$ is a line segment.
$\varepsilon \leftarrow +\infty$, **foreach** $\ell \in L$ **do** **if** $|\ell| < \varepsilon$ **then** $\varepsilon \leftarrow |\ell|$ where $|\ell|$ is the length of $\ell$.
**foreach** $b \in B$ **do** **foreach** $\ell \in L$ **where** $b \notin \ell$ **do** **if** $dist(b, \ell) < \varepsilon$ **then** $\varepsilon \leftarrow dist(b, \ell)$
**foreach** $q \in Q$ **do** **for** $i = 0$ *to* $k$ **do** **if** $dist(q, \overline{b_i b_{i+1}}) < \varepsilon$ **then** $\varepsilon \leftarrow dist(q, \overline{b_i b_{i+1}})$
$\varepsilon \leftarrow \varepsilon/(3\sqrt{2})$, $min \leftarrow +\infty$
Calculate the Minkowski sum of $L$ and an axis-parallel square $S$ with sides of length $2\varepsilon$ to obtain polygon $M$.
**foreach** $i \in \{0, k+1\}$ **do**
    **if** $i = 0$ **then** $j \leftarrow 1$, $l \leftarrow 2$ **else if** $i = k+1$ **then** $j \leftarrow k$, $l \leftarrow k-1$
    **if** *$b_i$ is an endpoint of exactly one segment $\overline{b_i q}$ in $L$ for some $q \in Q$* **then**
        $b'_i, b'_j \leftarrow$ vertices of $M$ within a distance $\sqrt{2}\varepsilon$ of $b_i$ and $b_j$, respectively, such that $|\overline{b'_i b'_j}|$ is minimized
    **else**
        **if** *$\overline{b_i b_j}$ is in $L$* **then**
            **if** *$p$ bends left in going from $\overline{b_i b_j}$ to $\overline{b_j b_l}$* **then** $dir \leftarrow$ clockwise **else** $dir \leftarrow$ counterclockwise
            $\ell_1 \leftarrow \overline{b_i b_j}$
            $\ell_2 \leftarrow$ segment in $L$ incident to $b_i$ next encountered in $dir$ radial sweep centered at $b_i$ from $\ell_1$
        **else**
            $\ell_1, \ell_2 \leftarrow$ line segments of $L$ incident to $b_i$ next encountered in clockwise and counterclockwise radial sweeps centered, respectively, at $b_i$ from $\overline{b_i b_j}$
        $e_1, e_2 \leftarrow$ edges of $M$ in the face of $M$ that corresponds to the face $f$ that are parallel to $\ell_1$ and $\ell_2$, respectively, such that $e_1$ is incident to $e_2$ or an endpoint of $e_1$ is adjacent to an endpoint of $e_2$ along $M$
        **if** *$e_1$ is incident to $e_2$* **then** $b'_i \leftarrow$ common endpoint of $e_1$ and $e_2$
        **else** $b'_i, b'_j \leftarrow$ vertices of $M$ within a distance $\sqrt{2}\varepsilon$ of $b_i$ and $b_j$, respectively, such that $|\overline{b'_i b'_j}|$ is minimized where $b'_i$ is also an endpoint of $e_1$ or $e_2$
Using $M$ as input, determine a Euclidean shortest path $p'$ connecting the points $b'_0$ and $b'_{k+1}$ in the face of $M$ that corresponds to face $f$ connecting the points $b'_0$ and $b'_{k+1}$.
Preappend the segment $\overline{b_0 b'_0}$ and append the segment $\overline{b_{k+1} b'_{k+1}}$ to the path $p'$.
**return** *path* $p'$

**Proof.** Frati [16] gave an algorithm that finds a SEFE for a forest and a planar graph without explicitly bounding the number of bends per edge. Here the pointset is dictated by the drawing of the planar graph and is used to simultaneously draw the forest using existing edges as fixed edges where applicable. We show how to draw an $n$-vertex planar graph $G$ on any pointset $P$ in general position of size $n$ such that each edge has at most $O(n)$ bends. Further, we allow for some edges initially to have been drawn as straight-line edges provided that the planar embedding of those edges still allows for some combinatorial embedding $\Gamma$ of $G$. We also specify the order in which the remaining edges are drawn as fixed edges and the corresponding faces in which to draw each edge, which together yield the embedding $\Gamma$ of $G$ after all the edges have been added.

The order of the edges is such that edges between disconnected components are drawn before edges connecting two vertices of the same component. If this were not the case, then an edge connecting two vertices in the same component might need to be routed around some disconnected components or isolated vertices that could later be connected to the rest of the graph by edges later in the sequence. This allows for each edge to be drawn with a Euclidean shortest path in the given face using the drawing algorithm DRAW-FIXED-EDGE. Here a Euclidean shortest path is the shortest simple curve in the Euclidean plane connecting two points so that no point of the curve intersects any existing line segment in the plane (except at endpoints). We will show that DRAW-FIXED-EDGE runs in $O(n^3)$ time to route the given edge around the existing $O(n^2)$ line segments that compose the $O(n)$ planar edges each with $O(n)$ bends. However, we need to ensure that an arbitrarily small distance $\varepsilon$ is maintained between the path and any points or line segments that correspond to vertices or to edges in the initial drawing with the exception of the endpoints of the path (as well as a small neighborhood around each of the two endpoints).

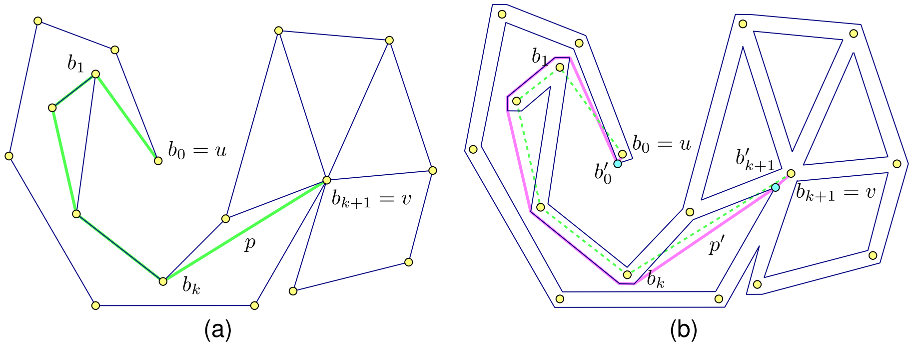


**Fig. 9.** An initial Euclidean shortest path $p$ from $u$ to $v$ in (a) is modified to the path $p'$ in (b) so that no bend or edge of $p$ (other than its endpoints) are within a distance $\varepsilon$ of any vertex or edge of the initial drawing.

Let $L$ be the $O(n^2)$ line segments that correspond to drawn edges $E'$ in a partial drawing of $G'(V, E')$ of $G(V, E)$, $\rho$ be the surjection of the line segments in $L$ to the edges in $E'$, and $Q$ be the $O(n^2)$ union of the $n$ points of $P$ with the $O(n^2)$ endpoints of the line segments in $L$. We wish to the draw edge $(u, v)$ in $E \setminus E'$ as a path in the given face $f$ of $G'$ with endpoints $\sigma(u)$ and $\sigma(v)$ in $P$. Using $\rho$, we determine the region $R$ enclosed by the line segments in $L$ that correspond to the edges of the face $f$ in line 4 of Draw-Fixed-Edge. This allows for us to compute a Euclidean shortest path $p$ from $\sigma(u)$ to $\sigma(v)$ confined to the region $R$ in line 5 of Draw-Fixed-Edge, which can be done in $O(|Q|\log|Q|) = O(n^2 \log n)$ time [21]; see Fig. 9(a). Let $B = \{b_0, \ldots, b_{k+1}\} \subseteq Q$ denote the $O(n)$ bends (which we will later show to be the case) of $p$ such that $\sigma(u) = b_0$, $\sigma(v) = b_{k+1}$, and $\overline{b_0 b_1}, \ldots, \overline{b_k b_{k+1}}$ are the line segments that form $p$. Note that $B$ only includes bends along the path $p$ that correspond to endpoints of segments in $L$, and does not include any other points in $P$ since $P$ is in general position.

Next we determine $\varepsilon$ to be the minimum of either (1) the length of any segment, computable in $O(|L|) = O(n^2)$ time, (2) the smallest distance between bends in $B$ and non-incident segments in $L$, computable in $O(|B||L|) = O(n^3)$ time, or (3) the smallest distance between points in $Q$ and segments in $p$, computable in $O(|B||Q|) = O(n^3)$ time, which is done in lines 8–10 of Draw-Fixed-Edge. We divide $\varepsilon$ by a factor of $3\sqrt{2}$ in line 11 of Draw-Fixed-Edge to ensure that when we take the Minkowski sum of the segments of $L$ with an axis-parallel square $S$ with sides of length $2\varepsilon$ (to obtain a polygon $M$ in line 12 of Draw-Fixed-Edge) we can guarantee that (i) no edge of $M$ is within a distance $\varepsilon$ of any segment in $L$ and (ii) no vertex of $M$ is within a distance $\varepsilon$ of any edge of $M$ that coincides with an edge of $p$. A factor of $\sqrt{2}$ gives (i) since the distance from the center of the square to a corner of the square is $\sqrt{2}\varepsilon$, which is the maximum distance from a segment of $L$ to an edge of $M$. An additional factor of 3 gives (ii) to allow the extra space on both sides of edges of $M$ as well as space in the middle to route future edges. Line 12 is easily accomplished in $O(|L|) = O(n^2)$ time, since the Minkowski sum of a non-convex set $L$ of complexity $O(n^2)$ with a convex square of complexity $m = 4$ is computed in $O(n^2 m)$ time [19]. Finally, we need to pick the two vertices $b'_0$ and $b'_{k+1}$ of $M$ (that correspond to the points $b_0$ and $b_{k+1}$) in the correct face of $M$ (that corresponds to the face $f$) of the segments of $L$ so that when we find a Euclidean shortest path $p'$ from $b'_0$ to $b'_{k+1}$ given $M$ as input in line 27 of Draw-Fixed-Edge in $O(|M|\log|M|) = O(n^2 \log n)$ time, the path $p'$ takes roughly the same route as $p$; see Fig. 9(b).

To ensure this, we have three distinct cases to consider where point $b_0$ (or $b_{k+1}$) is the endpoint of either

1. exactly one segment in $L$, i.e., corresponds to a degree-1 vertex in the graph,
2. two or more segments in $L$ where $\overline{b_0 b_1}$ (or $\overline{b_{k+1} b_k}$) is in $L$, or
3. two or more segments in $L$ where $\overline{b_0 b_1}$ (or $\overline{b_{k+1} b_k}$) is not in $L$.

*Case* 1: Here we pick $b'_0$ (or $b'_{k+1}$) and $b'_1$ (or $b'_k$) in line 16 of Draw-Fixed-Edge each to be one of the two or three vertices of $M$ that are within a distance of $\sqrt{2}\varepsilon$ of $b_0$ (or $b_{k+1}$) and $b_1$ (or $b_k$), respectively, such that the distance from $b'_0$ (or $b'_{k+1}$) to $b'_1$ (or $b'_k$) is minimized.

Cases 2 and 3 require us to determine the pair of line segments $\ell_1$ and $\ell_2$ incident to $b_0$ (or $b_{k+1}$) such that the line segment of path $p$ incident to $b_0$ (or $b_{k+1}$) lies between $\ell_1$ and $\ell_2$ in a radial sweep.

*Case* 2 (*determining* $\ell_1$ *and* $\ell_2$): We denote $\overline{b_0 b_1}$ (or $\overline{b_{k+1} b_k}$) as $\ell_1$ in line 20 of Draw-Fixed-Edge. First, we determine whether path $p$ starting from $b_0$ (or $b_{k+1}$) bends to the left or right in going from $\overline{b_0 b_1}$ (or $\overline{b_{k+1} b_k}$) to $\overline{b_1 b_2}$ (or $\overline{b_k b_{k-1}}$) in line 19 of Draw-Fixed-Edge. Next we do a radial sweep centered at $b_0$ (or $b_{k+1}$) proceeding in the opposite direction (i.e., if $p$ bends left, sweep clockwise, otherwise sweep counterclockwise) to determine the segment $\ell_2$ in $L$ incident to $b_0$ (or $b_{k+1}$) that the sweep next encounters in line 21 of Draw-Fixed-Edge, which takes $O(|M|\log|M|) = O(n^2 \log n)$ time.

*Case* 3 (*determining* $\ell_1$ *and* $\ell_2$): Here we determine $\ell_1$ and $\ell_2$ to be the two segments incident to $b_0$ (or $b_{k+1}$) that two radial sweeps centered at $b_0$ (or $b_{k+1}$) from $\overline{b_0 b_1}$ (or $\overline{b_{k+1} b_k}$) encounter in the clockwise and counterclockwise directions, respectively, in line 23 of Draw-Fixed-Edge, which also takes $O(n^2 \log n)$ time.

**Algorithm:** Draw-Remaining-Graph

▷ Given an $n$-vertex planar graph $G(V, E)$, a partial planar drawing $D$ of $G$ on a pointset $P$ in general position where any existing edges are straight-line segments, and an order in which to draw remaining edges and the faces in which to draw such edges (where edges connecting disconnecting components precede edges connecting vertices of the same component) so that the resulting embedding is planar, complete the drawing in $O(n^4)$ time using at most $O(n)$ bends per edge.

**Input**: Planar graph $G(V, E)$ on $n$ vertices, drawing $D$ of a subgraph $G'(V, E')$ on $n$ points in general position, bijection $\sigma : V \to P$, list $(e_1, \ldots, e_m)$ of $E_{remain} = E \setminus E'$, and corresponding list of faces $(f_1, \ldots, f_m)$ where face $f_i$ is in the graph $G_i(V, E \cup \{e_1, \ldots, e_{i-1}\})$ and contains both endpoints of $e_i$.

**Output**: Completed drawing $D$ of $G$

$P \leftarrow$ pointset used in $D$, $L \leftarrow$ existing line segments in $D$, $\rho \leftarrow$ surjection of $L \to E'$

**for** $i = 1$ *to* $m$ **do**

  $p \leftarrow \text{Draw} - \text{Fixed} - \text{Edge}(G_i, P, \sigma, L, \rho, e_i, f_i)$

  $L \leftarrow L \cup$ line segments of $p$, update $\rho : L \to E$

$D \leftarrow$ completed drawing of $G$ on final pointset $P$ and line segments $L$

**return** $D$

*Cases* 2 *and* 3 (*determining* $b'_0$ (*or* $b'_{k+1}$)): The edges $e_1$ and $e_2$ of $M$ that are parallel to $\ell_1$ and $\ell_2$, respectively, such that $e_1$ is incident to $e_2$ or an endpoint of $e_1$ is adjacent to an endpoint of $e_2$ can be determined in line 24 of Draw-Fixed-Edge in $O(|M|)$ time. If $e_1$ and $e_2$ are incident, we let $b'_0$ (or $b'_{k+1}$) be the common endpoint of $e_1$ and $e_2$ as in line 25 of Draw-Fixed-Edge. Otherwise, we let $b'_0$ (or $b'_{k+1}$) and $b'_1$ (or $b'_k$) be one of the $O(1)$ vertices of $M$ within a distance $\sqrt{2}\varepsilon$ from $b_0$ (or $b_{k+1}$) and $b_1$ (or $b_k$), respectively, such that $|\overline{b'_0 b'_1}|$ (or $|\overline{b'_k b'_{k+1}}|$) is minimized where $b'_0$ (or $b'_k$) is also an endpoint of $e_1$ or $e_2$ as in line 26 of Draw-Fixed-Edge.

The total running time of Draw-Fixed-Edge is $O(n^3)$ time provided that $|L| = O(n^2)$ and $|B| = O(n)$, which we now show to be the case. Clearly, $B \subseteq Q$, so $B$ is at most $O(|Q|)$. If $|L| = O(n)$, which is the case if all edges of $E'$ are drawn as straight-line segments, then clearly, $|B| = O(|Q|) = O(|P| + |L|) = O(n)$. However, each successive time that Draw-Fixed-Edge is called (assuming that the previous computed path $p'$ with $O(n)$ bends has been added to $L$), each bend in the newly computed path either occurs around one of the $n$ points in $P$ or around one of $O(n)$ bends of some fixed edge that was also routed around one of the $n$ points in $P$ in a previous call to Draw-Fixed-Edge. Each point in $P$ can be responsible for at most $O(n)$ bends over all the $O(n)$ edges, where each point contributes at most three bends to any given edge in the final drawing. This is because a Minkowski sum based upon an axis-parallel square is used to route edges so that if multiple edges are routed around a given point $p$, the one, two, or three bends of the respective edges caused by $p$ will nest inside each other. Hence, $|B| = O(n)$ and $|L| = O(n^2)$ since at most $O(n)$ edges with $O(n)$ bends can be added to $L$. The algorithm Draw-Remaining-Graph calls Draw-Fixed-Edge in this manner to construct the final planar drawing $D$ of $G$ with a total of $O(n^2)$ line segments in $\sum_{i=1}^{n} O(n^3) = O(n^4)$ time. □

Using the algorithm Draw-Remaining-Graph, we can now show how to compute a SEFE for any pair $(G, G')$, where $G \in \mathcal{P}_{\mathsf{SEFE}}$ and $G' \in \mathcal{P}$.

**Lemma 16.** *If $G$ is in $\mathcal{P}_{\mathsf{SEFE}}$, then it has a SEFE with any planar graph that can be computed in $O(n^4)$ time.*

**Proof.** Let $G_1 \in \mathcal{P}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{P}$. First, we find an embedding of $G_2$ and draw $G_2$ on an $(n-2) \times (n-2)$ grid, both done in $O(n)$ time [3,5]. Some of the edges of $G_1$ were drawn simultaneously with $G_2$. We can ignore the edges in $G_2 \setminus G_1$ as we draw the rest of $G_1$ using Draw-Remaining-Graph that runs in $O(n^4)$ time. The order in which the remaining edges of $G_1$ are drawn depends on the type of graph. For a forest or for a circular caterpillar in which the cycle has not yet been drawn, all the vertices are incident to the same face, and a shortest Euclidean path between any two vertices can always be found in that face. For a circular caterpillar with the cycle already drawn, the remaining points either lie inside or outside of the cycle. All edges are incident to the cycle. Hence, a Euclidean path always exists from vertices of the cycle to vertices of degree 1. Thus, for either forests or circular caterpillars, we first draw any edges of $G_1$ connecting disconnected components, and then draw the remaining cycle edge (if any) of $G_1$.

A graph with multiple cycles is either a $K_4$ or a subgraph of a $K_3$-multiedge that has two vertices $x$ and $y$ of degree greater than 2 such that $x$ and $y$ belong to each cycle in the graph. Let $C$ be a 4-cycle containing the vertices $x$ and $y$ (or a 3-cycle if there are no 4-cycles in $G_1$). After drawing edges connecting all isolated vertices to $x$ or $y$, we then finish drawing $C$ in either case. For $K_4$, one chord is then drawn inside of $C$, while the other chord is drawn outside of $C$. For a $K_3$-multiedge, any path from $x$ to $y$ is either the edge $(x, y)$ or the chain $x \hookrightarrow z \hookrightarrow y$ through some degree-2 vertex $z$. The edge $(x, y)$ is drawn inside of $C$. For the other chains $x \hookrightarrow z \hookrightarrow y$, there must always exist Euclidean paths from $x$ and $y$ to the common vertex $z$ that lies inside some cycle drawn so far. Any remaining edges must be incident to $x$ or $y$ in which a Euclidean path from $x$ to $y$ must also exist. □

Lemmas 13, 14, and 16 together imply the following characterization:

**Theorem 17.** *The following three statements are equivalent*:

- *A planar graph $G$ has a SEFE with any other planar graph.*
- *$G$ does not contain a subgraph homeomorphic to $K_3$ and a disjoint edge.*
- *$G$ is a forest, a circular caterpillar, $K_4$, or a subgraph of a $K_3$ multiedge.*

## 4. Characterizing pairs of outerplanar graphs having a simultaneous embedding with fixed edges

We next determine which biconnected outerplanar graphs *always* have a SEFE with *any* other outerplanar graph. A $K_3$-cycle is an $n$-cycle $C$ with chords such that every chord forms a 3-cycle with edges from $C$; see Fig. 2(e). Let $\mathcal{O}$ be the set of outerplanar graphs and $\mathcal{O}_{\mathsf{SEFE}}$ be the subset of $K_3$-cycles.

**Lemma 18.** *A biconnected outerplanar graph $G$ is in $\mathcal{O}_{\mathsf{SEFE}}$ if and only if $G$ does not have a subgraph homeomorphic to $G_{14,1}$.*

**Proof.** First, we prove necessity. Let $G$ be a $K_3$-cycle. For any pair of cycles $C$ and $C'$ in $G$ that share $(x, y)$ as a chord or as an edge, either $C$ or $C'$ is a 3-cycle since $x$ and $y$ are both adjacent to some vertex $z$. Hence, since $G_{14,1}$ consists of two 4-cycles that share a common chord of a 6-cycle, no subgraph of a $K_3$-cycle can have $G_{14,1}$ as a minor.

We prove sufficiency by showing that all biconnected outerplanar graphs in $\mathcal{O} \setminus \mathcal{O}_{\mathsf{SEFE}}$ have a subgraph homeomorphic to $G_{14,1}$. Since $G$ is biconnected, all the vertices must lie along the outerface $O$ that forms a simple cycle in $G$. If each chord $(x, y)$ has endpoints $x$ and $y$ for which there is a vertex $z$ whose incident edges are $(x,z)$ and $(y,z)$ then $G$ is a $K_3$-cycle. Otherwise, the two paths from $x$ to $y$ along the outerface $O$ must each have length of at least three, so that $O$ and the edge $(x, y)$ are homeomorphic to $G_{14,1}$. □

Corollary 12 and Lemma 18 are used to show the following lemma:

**Lemma 19.** *If $G$ is a biconnected outerplanar graph not in $\mathcal{O}_{\mathsf{SEFE}}$, then there is an outerplanar graph $G'$ that does not admit a SEFE with $G$.*

**Proof.** Let $G$ be any biconnected outerplanar graph in $\mathcal{O} \setminus \mathcal{O}_{\mathsf{SEFE}}$, and let $G'$ be any outerplanar graph in $\mathcal{O}$ such that $G'$ contains a subgraph homeomorphic to $G_{14,2}$. Of the 17 pairs from Theorem 10, only pair $(G_{14,1}, G_{14,2})$ consists of two biconnected outerplanar graphs; see Fig. 8(b). By Lemma 18, we know that $G$ has a subgraph homeomorphic to $G_{14,1}$. Thus, $(G, G')$ cannot have a SEFE by Corollary 12 since $G'$ has a subgraph homeomorphic to $G_{14,2}$. □

This shows that $G \in \mathcal{O}_{\mathsf{SEFE}}$ is a necessary condition for $G$ to have a SEFE with any outerplanar graph. To show that this condition is also sufficient, we need to show how to compute a SEFE for each $G \in \mathcal{O}_{\mathsf{SEFE}}$.

**Lemma 20.** *If $G$ is in $\mathcal{O}_{\mathsf{SEFE}}$, then it has a SEFE with any outerplanar graph that can be computed in $O(n^4)$ time.*

**Proof.** Let $G_1 \in \mathcal{O}_{\mathsf{SEFE}}$ and $G_2 \in \mathcal{O}$. We apply the algorithm Draw-Remaining-Graph in a similar manner as described in Lemma 16. First, we draw $G_2$ in $O(n)$ time using an outerplanar embedding of $G_2$ and ignore drawn edges in $G_2 \setminus G_1$ as we draw each of the remaining edges in $G_1$ using Draw-Fixed-Edge in $O(n^3)$ time. Recall that $G_1$ is a $K_3$-cycle. Let $C$ be the longest cycle in $G_1$. Such a cycle corresponds to the outerface of $G_1$ in its outerplanar embedding. All other edges are chords of $C$ that form $K_3$ subgraphs of $G_1$ with two edges of $C$. We note that $C$ will be the outerface of $G_1$ in the SEFE we compute so that $G_1$ will also have an outerplanar embedding.

First, we draw the remaining edges of $C$ that connect disconnected components of the graph, and then draw any remaining edges of $C$ proceeding in a clockwise direction along $C$ before drawing the chords of $C$ in an arbitrary order. Let $(x,z)$ be the next edge of $C$ to draw. While connecting disconnect components, all cycle edges are routed through the outerface guaranteeing that Draw-Fixed-Edge will produce a path for $(x,z)$ that does not create any more faces. Once the drawing is connected, any remaining cycle edge connects two vertices that are already connected in the graph, and hence, creates another face. There are two cases: (1) the edge $(x,z)$ closes the cycle $C$ (or a cycle $C'$ composed of cycle edges and chords of $C$) or (2) the edge $(x,z)$ is part of a $K_3$ subgraph $H$, where $(y,z)$, its companion edge along $C$, and $(x, y)$, the chord of $C$, have already been drawn. For case (1), we choose the outerface as the face in which to draw $(x,z)$ using Draw-Fixed-Edge. For case (2), if face $f$ of $C$ (or a subface $f'$ composed of a mixture of cycle edges and chords of $C$ that contains the vertices $x$, $y$, and $z$) has been drawn, then we also choose the outerface in which to draw $(x,z)$ so that the edge will not be drawn inside $f$ (of $f'$). Otherwise, we choose to draw $(x,z)$ to the right of both $(x, y)$ and $(y,z)$ (in the outerface) as we proceed clockwise around $C$ in drawing its edges. This guarantees that $G_1$ will have an outerplanar embedding. In drawing any remaining chord $(x, y)$, we choose the smallest subface $f'$ of $f$ that contains the cycle edges $(x,z)$ and $(y,z)$ in which to draw $(x, y)$ using Draw-Fixed-Edge. □

Lemmas 18, 19, and 20 together give the following characterization:

**Theorem 21.** *The following three statements are equivalent*:

- *A biconnected outerplanar graph $G$ has a* SEFE *with any other outerplanar graph.*
- *$G$ does not contain a subgraph homeomorphic to $G_{14,1}$.*
- *$G$ is a $K_3$-cycle.*

## 5. Conclusion

We gave a necessary condition for whether two graphs can have a SEFE in terms of 17 forbidden fixed edge minor pairs. This allowed us to characterize the graphs that always have a SEFE with any planar graph. We also characterized the class of biconnected outerplanar graphs that have a SEFE with any outerplanar graph. Our $O(n^4)$ time drawing algorithm with $O(n)$ bends per edge in Lemma 15 is an improvement over the SEFE drawing algorithm given in [16] for trees and planar graphs in which the number of bends per edge is unbounded.

While our results may be helpful in solving bigger open problems, there are still no known algorithms for testing whether a pair of planar graphs has a SEFE in polynomial time. It could be the case that there is only a constant number of forbidden minor pairs. Then, finding all fixed edge minor pairs of planar graphs would give a sufficient condition for their SEFE. Given the number of pairs to consider, an automated computer search may be the best approach. This may lead to a polynomial-time decision algorithm, an improvement over the ILP crossing minimization algorithm in [4].

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